On Translation-Surfaces Connected with a Unicursal Quartic.

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In a paper published in Vol. XXIX of the AMERICAN JOURNAL OF MATHEMATICS I have found and discussed all the types of algebraic translation-surfaces that can be generated in four different ways. Surfaces that admit of such fourfold generation were discovered by S. Lie, who in a series of papers* made known their general properties and method of analytical representation. A historical introduction to this interesting subject may be found in a paper published by Georg Scheffers in *Acta Mathematica*, Vol. 28, 1903, where also an independent treatment of certain parts of the theory is given.

With the exception of two theses by R. Kummer and Georg Wiegner† no detailed study of these surfaces has been undertaken, although, as G. Scheffers remarks,‡ such investigations promise sufficient results to justify the effort.

Owing to the large number of types of translation-surfaces admitting of fourfold generation, I limited myself in my former paper to the consideration of algebraic surfaces, reserving the investigation of transcendental surfaces to these and future investigations.

As is well known, all surfaces of this kind are closely connected with a quartic curve, irreducible or not, in the plane at infinity. All surfaces corresponding to projectively equivalent quartics are said to belong to the same *type*. It was found that *all algebraic surfaces correspond to a unicursal quartic having no double or triple points with distinct tangents.* (The correspondence here mentioned will be explained in what follows.)

The remaining unicursal quartics give rise to transcendental surfaces, the study of which is the object of the present paper.

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* *Berichte der Königlich. Säch. Gesells. der Wiss.*, 1896 and 1897. (See Bibliography.)

† Georg Wiegner, Dissertation. Leipzig, 1893.

‡ *Acta Math.*, vol. 28, 1904, p. 90.

Since the whole theory, according to Lie, is intimately bound up with Abel's theorem, the following pages may also be looked upon as a study of Abelian Integrals of the first kind with respect to a unicursal quartic.

The method of constructing translation-surfaces with a fourfold mode of generation is based on a theorem by Lie,* viz.:

If on a translation-surface that can be generated in more than two ways we draw tangents at any point along the four generating curves, the intersection of these tangents with the plane at infinity is a curve of the fourth order.

Conversely, if we suppose given in the plane at infinity a curve of the fourth order, there exist always infinitely many (∞^4) *surfaces generated in four ways, whose tangents along the generating curves cut the plane at infinity along the given curve.*

The coordinates of these surfaces are expressible as the sum of any two Abelian integrals with respect to the four points of intersection of a variable straight line with this quartic curve.

Every direction in space is determined by a point in the plane at infinity; the direction of a line joining a point to a consecutive point is determined whenever the ratios $\frac{dx}{dz}$ and $\frac{dy}{dz}$ are given. We may therefore, with Lie, consider these ratios as coordinates ξ, η in the plane at infinity.

Let there be given in this plane a quartic curve $F(\xi, \eta) = 0$; in order to determine the translation-surface, according to Lie's theorem, we form the Abelian integrals

$$\Phi = \int \frac{\xi d\xi}{F'_{(\eta)}}, \quad \Psi = \int \frac{\eta d\xi}{F'_{(\eta)}}, \quad \mathrm{X} = \int \frac{d\xi}{F'_{(\eta)}},$$

whose limits we fix as follows: We suppose the quartic cut by a fixed and a variable straight line; denoting the abscissas of the point of intersection by ξ_1^0, ξ_2^0, ξ_3^0, ξ_4^0 and ξ_1, ξ_2, ξ_3, ξ_4 respectively, we choose the former as the lower and the latter as the upper limits, so that we have

$$\Phi_i = \int_{\xi_i^0}^{\xi_i} \frac{\xi_i d\xi_i}{F'_{(\eta_i)}}, \quad \Psi_i = \int_{\xi_i^0}^{\xi_i} \frac{\eta_i d\xi_i}{F'_{(\eta_i)}}, \quad \mathrm{X}_i = \int_{\xi_i^0}^{\xi_i} \frac{d\xi_i}{F'_{(\eta_i)}}.$$

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* *Berichte der Säch. Gesells. der Wiss.*, vol. 48, 1896, p. 197.

Now by Abel's theorem we have (the constants ξ_i^0 being properly chosen):

$$\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \equiv 0,$$
$$\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 \equiv 0,$$
$$X_1 + X_2 + X_3 + X_4 = 0,$$

from which it follows that

$$\Phi_1 + \Phi_2 \equiv -\Phi_3 - \Phi_4,$$
$$\Psi_1 + \Psi_2 \equiv -\Psi_3 - \Psi_4,$$
$$X_1 + X_2 \equiv - X_3 - X_4,$$

so that the equations

$$x = \Phi_1 + \Phi_2, \qquad y = \Psi_1 + \Psi_2, \qquad z = X_1 + X_2$$

represent the same surface as

$$x = -\Phi_3 - \Phi_4, \qquad y = -\Psi_3 - \Psi_4, \qquad z = -X_3 - X_4,$$

a translation-surface generated in four ways, as is seen from the double mode of representation.

If the quartic is irreducible, the integrals Φ_i have the same form; the same is true of the Ψ's and X's. The curves ξ_1 and ξ_2 cover the surface twice and are all parallel to each other and similarly placed. The curve $\xi_1 = \xi_2$ is a special asymptotic line on the surface and the envelope of the curves $\xi_1 = \text{const.}$, $\xi_2 = \text{const.}$ The same is true of the curves ξ_3 and ξ_4 which have for envelope the special asymptotic line $\xi_3 = \xi_4$. The surface may also be considered as the locus of the middle points of all chords of the curve $\xi_1 = \xi_2$ or of the curve $\xi_3 = \xi_4$. It should be noticed that the surface is symmetric with respect to a certain point which, by properly fixing the lower limits of the integrals, may be taken as the origin; *it has therefore a center.*

I.

We shall begin with a quartic having three non-consecutive double points; by a projective transformation (real or imaginary) the curve may be thrown into the form, using x, y instead of ξ, η,

$$x^2 + y^2 - 2axy + x^2y^2 - 2bx^2y - 2cxy^2 = 0, \tag{1}$$

in which the double points are placed at the vertices of the triangle of reference. In order to find a suitable parametric representation we intersect the curve by the hyperbola

$$xy + \rho x + \sigma y = 0, \tag{2}$$

which passes through the double points; let it also pass through the point of

intersection of $y = mx$ with the curve, m being one of the roots of the equation $m^2 - 2am + 1 = 0$. This point is easily found to be $\frac{2(b+mc)}{m}$, $2(b+mc)$. Now in order that the hyperbola (2) shall pass through this point, the following relation between ρ and σ must exist:

$$m\sigma + \rho = -2(b+mc).$$

Substituting the value of x from (2) in (1) we have

$$(\sigma^2 + 2c\sigma + 1)\,y^2 + 2(\rho + a\sigma - b\sigma^2 + c\sigma\rho)\,y + \sigma^2 + \rho^2 + 2a\sigma\rho = 0,$$

of which $y + m\sigma + \rho$ is a factor. There remains therefore, after dividing the expression,

$$(\sigma^2 + 2c\sigma + 1)\,y + \rho + (2a - m)\,\sigma,$$

which gives us the required parametric representation:

$$y = \frac{-\left(\frac{\sigma}{m} + \rho\right)}{\sigma^2 + 2c\sigma + 1} = \frac{(1-m^2)\,\rho + 2(b+mc)}{\rho^2 + (4b + 2mc)\,\rho + m^2 + 4b^2 + 4bmc},$$

$$x = \frac{-(\sigma + m\rho)}{\rho^2 + 2b\rho + 1} = \frac{(1-m^2)\,\rho + 2(b+mc)}{m(\rho^2 + 2b\rho + 1)}.$$

We also find

$$dx = \frac{(1-m^2)\,\rho^2 + 4(b+mc)\,\rho + 4b(b+mc) + m^2 - 1}{(\rho^2 + 2b\rho + 1)^2}\,d\rho$$

and

$$F'_{(y)} = \frac{2\sigma y}{(y+\rho)^2}\left[(b\sigma - a)\,y + \sigma + a\rho\right]$$

$$= \frac{\sigma^2 y}{(y+\rho)^2}\left[\frac{(1-m^2)\,\rho^2 + 4(b+mc)\,\rho + 4b(b+mc) + m^2 - 1}{\sigma^2 + 2c\sigma + 1}\right],$$

$$\frac{dx}{F'_{(y)}} = \frac{-(y+\rho)^2(\sigma^2 + 2c\sigma + 1)}{\sigma^2\,y\,(\rho^2 + 2b\rho + 1)} = \frac{(y+\rho)^2\left(\frac{\sigma}{m} + \rho\right)}{\sigma^2\,y^2\,(\rho^2 + 2b\rho + 1)^2};$$

but

$$x^2 = \frac{\sigma^2\,y^2}{(y+\rho)^2} = \frac{(\sigma + m\rho)^2}{(\rho^2 + 2b\rho + 1)^2},$$

hence

$$\left.\begin{aligned}
\frac{dx}{F'_{(y)}} &= -\frac{d\rho}{(1-m^2)\,\rho + 2(b+mc)},\\
\frac{xdx}{F'_{(y)}} &= -\frac{d\rho}{m(\rho^2 + 2b\rho + 1)},\\
\frac{ydx}{F'_{(y)}} &= -\frac{d\rho}{\rho^2 + (4b + 2mc)\,\rho + m^2 + 4b^2 + 4bmc}.
\end{aligned}\right\}\quad (3)$$

The surface may now be written, putting

$$-mX = X', \qquad -Y = Y', \qquad (m^2 - 1)\,Z = Z'$$

and dropping the primes,

$$\begin{aligned}
X &= \int \frac{d\rho_1}{\rho_1^2 + 2b\rho_1 + 1} + \int \frac{d\rho_2}{\rho_2^2 + 2b\rho_2 + 1}, \\
Y &= \int \frac{d\rho_1}{\rho_1^2 + (4b + 2mc)\,\rho_1 + m^2 + 4b^2 + 4bmc} \\
&\qquad + \int \frac{d\rho_2}{\rho_2^2 + (4b + 2mc)\,\rho_2 + m^2 + 4b^2 + 4bmc}, \\
Z &= \int \frac{d\rho_1}{\rho_1 + \dfrac{2\,(b + mc)}{1 - m^2}} + \int \frac{d\rho_2}{\rho_2 + \dfrac{2\,(b + mc)}{1 - m^2}}.
\end{aligned} \tag{3'}$$

The discriminants of the three quadratic equations

$$\begin{aligned}
&\rho_1^2 + 2b\rho_1 + 1 = 0, \\
&\rho_1^2 + (4b + 2mc)\,\rho_1 + m^2 + 4b^2 + 4bmc = 0, \\
&m^2 - 2am + 1 = 0
\end{aligned} \tag{4}$$

are $b^2 - 1$, $m^2(c^2 - 1)$, $a^2 - 1$, respectively. If therefore b, c and a be greater than unity, the integration will give rise to logarithmic functions. In case either b, c (or both) is less than unity, while a is greater than unity, antitrigonometric functions instead of logarithmic will be introduced in the equation of the surface. These cases will be discussed later. If a is less than unity the surface (3) is apparently imaginary, although in reality it is as real as in the three preceding cases; this case, therefore, needs separate treatment. For the present we need not distinguish between the different cases; we shall integrate without regard to the sign of the discriminants of (4), it being understood that whenever b, c, or both, are less than unity the surface may be thrown into a real form by introducing trigonometric functions in the coordinates X and Y. This remark is also applicable to the case where all three parameters are less than unity, but, as we have said before, a separate treatment is needed. The geometric interpretation of each of the four cases will also be explained hereafter.

Calling the roots of the first two equations (4) α_1, β_1; α_2, β_2 respectively, we have after integrating:

$$X = \frac{1}{2\sqrt{b^2-1}} \log \frac{(\rho_1-\alpha_1)(\rho_2-\alpha_1)}{(\rho_1-\beta_1)(\rho_2-\beta_1)},$$

$$Y = \frac{1}{2\sqrt{c^2-1}} \log \frac{(\rho_1-\alpha_2)(\rho_2-\alpha_2)}{(\rho_1-\beta_2)(\rho_2-\beta_2)},$$

$$Z = \log(\rho_1-k_1)(\rho_2-k_1), \quad k_1 = -\frac{2(b+mc)}{1-m^2}.$$

By using the transformation $2\sqrt{b^2-1}\,X = X'$, $2\sqrt{c^2-1}\,Y = Y'$, $Z = Z'$, these equations may be written:

$$e^X = \frac{(\rho_1-\alpha_1)(\rho_2-\alpha_2)}{(\rho_1-\beta_1)(\rho_2-\beta_1)}, \quad e^Y = \frac{(\rho_1-\alpha_2)(\rho_2-\alpha_2)}{(\rho_1-\beta_2)(\rho_2-\beta_2)}, \quad e^Z = (\rho_1-k_1)(\rho_2-k_1),$$

or,

$$\begin{gathered} \rho_1\rho_2 - k_1(\rho_1+\rho_2) + k_1^2 - e^Z = 0, \\ (1-e^X)\rho_1\rho_2 - (\alpha_1-\beta_1 e^X)(\rho_1+\rho_2) + \alpha_1^2 - \beta_1^2 e^X = 0, \\ (1-e^Y)\rho_1\rho_2 - (\alpha_2-\beta_2 e^Y)(\rho_1+\rho_2) + \alpha_2^2 - \beta_2^2 e^Y = 0, \end{gathered}$$

from which by elimination we obtain

$$\begin{vmatrix} 1 & -k_1 & k_1^2 - e^Z \\ 1-e^X & -(\alpha_1-\beta_1 e^X) & \alpha_1^2 - \beta_1^2 e^X \\ 1-e^Y & -(\alpha_2-\beta_2 e^Y) & \alpha_2^2 - \beta_2^2 e^Y \end{vmatrix} = 0,$$

which expanded may be written

$$A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Y+Z} + He^{X+Y+Z} = 0, \qquad (5)$$

where the coefficients have the following values:

$$\begin{aligned} A &= (\alpha_1-\alpha_2)\,[\alpha_1\alpha_2 - k_1(\alpha_1+\alpha_2) + k_1^2], \\ B &= (\alpha_2-\beta_1)\,[\alpha_2\beta_1 - k_1(\alpha_2+\beta_1) + k_1^2], \\ C &= (\beta_2-\alpha_1)\,[\alpha_1\beta_2 - k_1(\beta_2+\alpha_1) + k_1^2], \\ D &= \alpha_2 - \alpha_1, \\ E &= \beta_1 - \alpha_2, \\ F &= (\beta_1-\beta_2)\,[\beta_1\beta_2 - k_1(\beta_1+\beta_2) + k_1^2], \\ G &= \alpha_1 - \beta_2, \\ H &= \beta_2 - \beta_1. \end{aligned}$$

These coefficients are not independent; in fact, the following identical relation is easily seen to exist between them:

$$EGAF = HDCB, \tag{6}$$

which is of fundamental importance.

We have then the

THEOREM: *To a unicursal quartic having three double points with distinct tangents there corresponds a translation-surface of the form*

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with the following identical relation between the coefficients:

$$EGAF = HDCB. \tag{6}$$

There exist ∞^3 *types of such surfaces corresponding to the* ∞^3 *projectively non-equivalent quartics with three non-consecutive double points.*

Every surface (5) has a center which is found by putting $X = X' - \xi$, $Y = Y' - \eta$, $Z = Z' - \zeta$ in (5). After this transformation the new coefficients $A, B', C', \ldots, H'$ must satisfy the following conditions:

$$A = -H', \; B' = -G', \; C' = -E', \; D' = -F';$$

we find then the following equalities:

$$\xi + \eta + \zeta = \log\left(\frac{H}{A}\right),$$

$$\eta + \zeta - \xi = \log\left(\frac{G}{B}\right),$$

$$\zeta + \xi - \eta = \log\left(\frac{E}{C}\right),$$

$$\xi + \eta - \zeta = \log\left(\frac{F}{D}\right).$$

Solving the first three equations, we have

$$\xi = \tfrac{1}{2}\log\left(\frac{BH}{AG}\right), \quad \eta = \tfrac{1}{2}\log\left(\frac{CH}{AE}\right), \quad \zeta = \tfrac{1}{2}\log\left(\frac{EG}{BC}\right), \tag{7}$$

which values are found to satisfy the fourth equation, owing to the relation (6). The surface has now the following simple form:

$$A(1 - e^{X+Y+Z}) + B'(e^{Y+Z} - e^X) + C'(e^{X+Z} - e^Y) + D'(e^{X+Y} - e^Z) = 0, \tag{8}$$

whose center of symmetry is at the origin.

Remark. A translation $X = X' - 2n\pi i$, $Y = Y' - 2n\pi i$, $Z = Z' - 2n\pi i$, where n is any positive or negative integer, leaves the surface invariant, while a translation $X = X' - n\pi i$, $Y = Y' - n\pi i$, $Z = Z' - n\pi i$ transforms it into a real surface whose center of symmetry is at the point $n\pi i$, $n\pi i$, $n\pi i$, viz.:

$$A(1 + e^{X+Y+Z}) + B'(e^{Y+Z} + e^{X}) + C'(e^{X+Z} + e^{Y}) + D'(e^{X+Y} + e^{Z}) = 0. \quad (9)$$

Before proceeding further we shall introduce a few definitions due to Lie:

If we transform a twisted curve in the space (x, y, z) by the transformation

$$x_1 = \lambda x, \quad y_1 = \mu y, \quad z_1 = \nu z, \quad (10)$$

we obtain a family of ∞^3 curves which evidently remains invariant for the transformation. We say then that *these curves belong to the same species* (Gattung). The same definition may also be extended to surfaces.*

Another fruitful idea due to Lie is the logarithmic transformation:

$$X = \log x, \quad Y = \log y, \quad Z = \log z, \quad (11)$$

where (x, y, z) is the so-called logarithmic space.†

Consider now all the curves of the same species,

$$x = \lambda . \phi(t), \quad y = \mu . \psi(t), \quad z = \nu . \chi(t);$$

transforming by (11), we have

$$X = \log \phi(t) + \log \lambda, \quad Y = \log \psi(t) + \log \mu, \quad Z = \log \chi(t) + \log \nu,$$

by which we obtain in the space (X, Y, Z) all the curves that are parallel to each other and similarly placed. Hence:

To all the ∞^3 curves in space (X, Y, Z) obtained by the ∞^3 translations of a twisted curve there correspond in the space (x, y, z) all the ∞^3 curves of the same species. This is also evident from the fact that to a translation in (X, Y, Z) corresponds the affinity transformation (10).

Moreover, to the transformation $X = -X$, $Y = -Y$, $Z = -Z$ (the so-called reflexion, *Spieglung*) corresponds the involutary transformation

$$x_1 = \frac{1}{x}, \quad y_1 = \frac{1}{y}, \quad z_1 = \frac{1}{z}. \quad (12)$$

The general involutary transformation

$$x_1 = \frac{\lambda}{x}, \quad y_1 = \frac{\mu}{y}, \quad z_1 = \frac{\nu}{z} \quad (13)$$

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* See Lie-Scheffers, *Berühr. Trans.*, pp. 331 and 334.

† *Ibid.*, p. 356.

may be considered as a succession of the transformations (10) and (12), so that we may say:

To the general involutary transformation (13) *in the space* (x, y, z) *there corresponds a reflexion of all the points of* (X, Y, Z) *with respect to the point* $(\log \lambda, \log \mu, \log \nu)$.

If now we apply the logarithmic transformation to the surface (5), we obtain the cubic surface

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \tag{14}$$

the coefficients of which satisfy the same relation as before, viz.:

$$EGAF = HDCB. \tag{6}$$

These ∞^3 surfaces remain invariant by the involutary transformation (13). The transformed surface is:

$$Axyz + \lambda Byz + \mu Cxz + \nu Dxy + \lambda\nu Ey + \mu\lambda Fz + \nu\mu Gx + \lambda\mu\nu H = 0,$$

which is evidently of the same form as (14) with the same relation (6) between the coefficients.

From the above it is easily seen that there is one set of values of λ, μ, ν which will leave the surface invariant, viz.:

$$\lambda = \frac{AG}{BH}, \quad \mu = \frac{EA}{CH}, \quad \nu = \frac{AF}{HD}, \tag{15}$$

as is easily verified, taking into account the identical relation (6). To the translation curves on (5) correspond the double set of twisted cubics on the surface represented by the equations

$$\begin{aligned} x &= \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}, \\ y &= \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}, \\ z &= (\rho_1 - k_1)(\rho_2 - k_1). \end{aligned} \tag{16}$$

The curves $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$ constitute a family of curves of the same species which cover the surface doubly and may therefore be considered rather as two families, both made up of curves of the same species. By means of the involutary transformation

$$x_1 = \frac{\lambda}{x}, \quad y_1 = \frac{\mu}{y}, \quad z_1 = \frac{\nu}{z}, \tag{13}$$

where λ, μ, ν have the values given in (15), we obtain the same surface, but in another analytic form, viz.:

$$x_1 = \lambda \frac{(\rho_3 - \beta_1)(\rho_4 - \beta_1)}{(\rho_3 - \alpha_1)(\rho_4 - \alpha_1)},$$
$$y_1 = \mu \frac{(\rho_3 - \beta_2)(\rho_4 - \beta_2)}{(\rho_3 - \alpha_2)(\rho_4 - \beta_2)}, \tag{17}$$
$$z_1 = \frac{\nu}{(\rho_3 - k_1)(\rho_4 - k_1)},$$

on which the curves ρ_3 and ρ_4 are two families of the same species. We thus see that the involutary transformation (13) has transformed the curves $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$, into $\rho_3 = \text{const.}$, $\rho_4 = \text{const.}$, each pair of families belonging to the same species; while any two curves belonging to different pairs are of different species. We may therefore say:

The surface

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0 \tag{18}$$

contains two pairs of families of curves, each pair consisting of curves of the same species. The surface may be generated by performing on any one of these curves ∞^1 *affinity transformations; that is, the surface admits of a fourfold mode of generation.**

This surface is thus seen to be analogous to the surface

$$Ayz + Bzx + Cxy + Lx + My + Nz = 0,$$

which, as S. Lie has shown,† has a similar mode of generation; it has four families of curves; viz., the two sets of generators and two families of cubic curves. By the inverse of the logarithmic transformation this surface is transformed into a translation-surface

$$Ae^{Y+Z} + Be^{Z+X} + Ce^{X+Y} + Le^X + Me^Y + Ne^Z = 0,$$

which corresponds to the case where the quartic degenerates into two intersecting conics.‡ Whenever this happens the curves belonging to either pair (a set of generators and a family of cubics) *are not of the same species*; this is due to the fact that since the surface (X, Y, Z) corresponds to a degenerate quartic (two conics), the functions Φ_1 and Φ_3 (see p. 172) are of identically the same form, and

* By fourfold mode of generation we mean in this case that the same surface may be represented in two different ways, namely (16) and (17). The phrase *fourfold mode* applies here to the logarithmic space (x, y, z).

† G. Scheffers, *Berühr. Trans.*, Vol. I, pp. 350 and 364.

‡ *Ibid.*

likewise Φ_2 and Φ_4, while Φ_1 and Φ_2 and also Φ_3 and Φ_4 are not; the same is also true of the Ψ's and X's.

Conversely, let the surface (17) be given. Since we know that it contains two pairs of families of curves, each pair being of the same species, and that either pair by the reflexion (13) is transformed into the other, we conclude that *the surface*

$$A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Y+Z} + He^{X+Y+Z} = 0 \qquad (18)$$

is a translation-surface containing two pairs of families of translation-curves, and thus admits of a fourfold generation.

Let the surface (18) be referred to its center of symmetry as origin, writing it as before

$$A\,(1 - e^{X+Y+Z}) + B'\,(e^{Y+Z} - e^X) + C'\,(e^{X+Y} - e^Z) + D'\,(e^{X+Z} - e^Y) = 0. \qquad (8)$$

Putting $X = -X$, $Y = -Y$, $Z = -Z$ and subtracting the result from (8), we have

$$A\,(e^{-(X+Y+Z)} - e^{X+Y+Z}) + B'\,(e^{-(Y+Z)} - e^{Y+Z} + e^{-X} - e^X)$$
$$+ C'\,(e^{-(X+Y)} - e^{X+Y} + e^{-Z} - e^Z) + D'\,(e^{-(X+Z)} - e^{X+Z} - e^Y + e^{-Y}) = 0.$$

If now we employ the transformation $X = i\,X_1$, $Y = i\,Y_1$, $Z = i\,Z_1$, and reduce, this equation takes the form

$$A \sin(X_1 + Y_1 + Z_1) + B'\,[\sin(Y_1 + Z_1) - \sin X_1] + C'\,[\sin(X_1 + Y_1) - \sin Z_1]$$
$$+ D'\,[\sin(X_1 + Z_1) - \sin Y_1] = 0,$$

which again reduces to

$$A \sin \tfrac{1}{2}(X_1 + Y_1 + Z_1) + B' \sin \tfrac{1}{2}(Y_1 + Z_1 - X_1) + C' \sin \tfrac{1}{2}(X_1 + Y_1 - Z_1)$$
$$+ D' \sin \tfrac{1}{2}(X_1 + Y_1 - Z_1) = 0,$$

and finally, putting $\tfrac{1}{2}X_1 = X$, $\tfrac{1}{2}Y_1 = Y$, $\tfrac{1}{2}Z_1 = Z$,

$$A \sin(X + Y + Z) + B' \sin(Y + Z - X) + C' \sin(X + Y - Z)$$
$$+ D' \sin(X + Z - Y) = 0. \qquad (19)$$

This transformation, it will be noticed, has no effect on the corresponding quartic in the plane at infinity. In the new space $(i\,X,\ i\,Y,\ i\,Z)$ the surface appears as a real surface with three real periods, while in the original space it had three imaginary periods. They both belong to the same type, provided a, b and c in the quartic have constant values. They are, moreover, very different in form: the surface (19) is contained in a cube whose side equals π, and the whole of space being divided into such cubes, each one contains an exact reproduction

the surface in the original cube. The surface (8) shows no such periodicity, the periods being imaginary. It is thus seen that *leaving the quartic curve in the plane at infinity invariant, we can express the corresponding surface either as a surface having imaginary periods, or as one having real periods.*

We may express the above results in the following

THEOREM: *To a unicursal quartic having three non-consecutive double points with distinct tangents there corresponds a translation-surface of the form*

$$A + Be^{X} + Ce^{Y} + De^{Z} + Ee^{X+Z} + Fe^{X+Y} + Ge^{Y+Z} + He^{X+Y+Z} = 0,$$

with the following identical relation between the coefficients:

$$E\,G\,A\,B = H\,D\,C\,B.$$

The surface, when transformed to its center of symmetry as origin, takes the form

$$A\,(1 - e^{X+Y+Z}) + B'\,(e^{Y+Z} - e^{X}) + C'\,(e^{X+Y} - e^{Z}) + D'\,(e^{X+Z} - e^{Y}) = 0,$$

which by means of the transformation

$$X = 2\,i\,X_1, \quad Y = 2\,i\,Y_1, \quad Z = 2\,i\,Z_1,$$

may be put into the form

$$A \sin (X_1 + Y_1 + Z_1) + B' \sin (Y_1 + Z_1 - X_1) + C' \sin (X_1 + Y_1 - Z_1) \\ + D' \sin (X_1 + Z_1 - Y_1) = 0.$$

Remark. If in (19) we put $Y + Z - X = X_1$, $X + Y - Z = Y_1$, $X + Z - Y = Z_1$, the equation becomes

$$A \sin (X_1 + Y_1 + Z_1) + B' \sin X_1 + C' \sin Y_1 + D' \sin Z_1 = 0,$$

which for certain purposes may be simpler and more convenient.

II.

In the case where the three double points have imaginary pairs of tangents (the three vertices of the triangle being conjugate points), the parametric representation of the quartic that we have used (p. 173) becomes inconvenient, if we want the surface in a real form; in fact, k_1 becomes imaginary with m, since m is a root of the equation $m^2 - 2\,a\,m + 1 = 0$, a now being less than unity. To avoid this difficulty we must find a suitable parametric representation.

We write the quartic as before,

$$x^2 + y^2 - 2\,a\,x\,y + x^2\,y^2 - 2\,b\,x^2\,y - 2\,c\,x\,y^2 = 0, \tag{1}$$

or,

$$\frac{1}{x^2} + \frac{1}{y^2} - \frac{2a}{x\,y} + 1 - \frac{2b}{y} - \frac{2c}{x} = 0, \tag{2}$$

from which it is seen that a parametric representation of (1) may be found by obtaining one for the conic

$$x_1^2 + y_1^2 - 2\,a\,x_1 y_1 - 2\,b\,y_1 - 2\,c\,x_1 + 1 = 0$$

(see Salmon's Higher Plane Curves, p. 244 *), obtained by putting $x_1 = \frac{1}{x}$, $y_1 = \frac{1}{y}$, $z_1 = \frac{1}{z}$ in (2). Since a, b and c are all less than unity, this conic (in general an ellipse) lies wholly inside the triangle of reference. Transforming the origin to the center $\left(\frac{c + a\,b}{1 - a^2}, \frac{b + a\,c}{1 - a^2}\right)$, we have

$$\bar{x}_1^2 + \bar{y}_1^2 - 2\,a\,\bar{x}_1 \bar{y}_1 = \frac{a^2 + b^2 + c^2 + 2\,a\,b\,c - 1}{1 - a^2} = \frac{R^2}{1 - a^2},$$

from which it appears that the ellipse, and hence the quartic, is real whenever R^2 is positive. In order to express $\bar{x}_1$ and $\bar{y}_1$ in terms of a variable parameter ρ, we pass a line $y = \rho\,x + \sigma$ through the point $\left(\frac{R}{\sqrt{1 - a^2}}, 0\right)$ and find the second and variable point of intersection, which is

$$\bar{x}_1 = \frac{(\rho_1^2 - 1)\,R}{\sqrt{1 - a^2}\,(1 - 2\,a\,\rho + \rho^2)}, \quad \bar{y}_1 = \frac{2\,\rho\,(a\,\rho - 1)\,R}{\sqrt{1 - a^2}\,(1 - 2\,a\,\rho + \rho^2)};$$

and hence,

$$x_1 = \bar{x}_1 + h = \frac{(\rho^2 - 1)\,R}{\sqrt{1 - a^2}\,(1 - 2\,a\,\rho + \rho^2)} + \frac{c + a\,b}{1 - a^2},$$

$$y_1 = \bar{y}_1 + k = \frac{2\,\rho\,(a\,\rho - 1)\,R}{\sqrt{1 - a^2}\,(1 - 2\,a\,\rho + \rho^2)} + \frac{b + a\,c}{1 - a^2},$$

so that we finally have the following values for x and y on the quartic:

$$\left.\begin{aligned} x &= \frac{(1 - a^2)\,(1 - 2\,a\,\rho + \rho^2)}{[\sqrt{1 - a^2}\,R + c + a\,b]\,\rho^2 - 2\,a\,(c + a\,b)\,\rho + c + a\,b - \sqrt{1 - a^2}\,R} \\ y &= \frac{(1 - a^2)\,(1 - 2\,a\,\rho + \rho^2)}{[2\sqrt{1 - a^2}\,a\,R + b + a\,c]\,\rho^2 - [2\,a\,(b + a\,c) + 2\sqrt{1 - a^2}\,R]\,\rho + b + a\,c} \end{aligned}\right\} \quad (2)$$

We also have

$$F'_{(y)} = y - 2\,c\,x\,y + y\,x^2 - a\,x - b\,x\,y = x\sqrt{(b^2 - 1)\,x^2 + 2\,(c + a\,b)\,x + a^2 - 1}$$

and

$$\frac{dx}{d\rho} = \frac{2\,(1 - a^2)\sqrt{1 - a^2}\,R\,(a\,\rho^2 - 2\,\rho + 1)}{[(\sqrt{1 - a^2}\,R + c + a\,b)\,\rho^2 - 2\,a\,(c + a\,b)\,\rho + c + a\,b - \sqrt{1 - a^2}\,R]^2}.$$

* We refer here to the second edition of this work.

By substituting in $F'_{(y)}$ the value of x in terms of ρ, we have

$$F'_{(y)}=\frac{x\sqrt{1-a^2}\,R\,(a\rho^2-2\rho+a)}{(\sqrt{1-a^2}\,R+c+ab)\rho^2-2a(c+ab)\rho+c+ab-\sqrt{1-a^2}\,R}.$$

The corresponding surface may now be written:

$$\left.\begin{aligned}
X&=2(1-a^2)\int\frac{d\rho_1}{D_1\rho_1}+2(1-a^2)\int\frac{d\rho_2}{D_1\rho_2},\\
Y&=2(1-a^2)\int\frac{d\rho_1}{D_2\rho_1}+2(1-a^2)\int\frac{d\rho_2}{D_2\rho_2},\\
Z&=2\int\frac{d\rho_1}{1-a\rho_1+\rho_1^2}+2\int\frac{d\rho_2}{1-a\rho_2+\rho_2^2},
\end{aligned}\right\}\quad(3)$$

where D_1 and D_2 are the respective denominators of x and y in (2). It should be observed that the discriminant of D_1 and D_2, viz.: $(1-a^2)^2(b^2-1)$ and $(1-a^2)^2(c^2-1)$ respectively, are both negative, b and c being less than unity. We have now, after integrating and transforming in a suitable manner to get rid of extraneous factors,

$$\left.\begin{aligned}
X&=\tan^{-1}\frac{\rho_1-\dfrac{a(c+ab)}{\sqrt{1-a^2}\,R+c+ab}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{\sqrt{1-a^2}\,R+c+ab}}+\tan^{-1}\frac{\rho_2-\dfrac{a(c+ab)}{\sqrt{1-a^2}\,R+c+ab}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{\sqrt{1-a^2}\,R+c+ab}}\\
Y&=\tan^{-1}\frac{\rho_1-\dfrac{a(b+ac)+\sqrt{1-a^2}\,R}{2\sqrt{1-a^2}\,aR+b+ac}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{2\sqrt{1-a^2}\,aR+b+ac}}+\tan^{-1}\frac{\rho_2-\dfrac{a(b+ac)+\sqrt{1-a^2}\,R}{2\sqrt{1-a^2}\,aR+b+ac}}{\dfrac{(1-a^2)\sqrt{1-b^2}}{2\sqrt{1-a^2}\,aR+b+ac}}\\
Z&=\tan^{-1}\frac{\rho_1-a}{\sqrt{1-a^2}}+\tan^{-1}\frac{\rho_2-a}{\sqrt{1-a^2}}.
\end{aligned}\right\}\quad(4)$$

In order to facilitate elimination we write these equations in the form

$$\left.\begin{aligned}
X&=\tan^{-1}\frac{\rho_1-\alpha_1}{k_1}+\tan^{-1}\frac{\rho_2-\alpha_1}{k_1},\\
Y&=\tan^{-1}\frac{\rho_1-\alpha_2}{k_2}+\tan^{-1}\frac{\rho_2-\alpha_2}{k_2},\\
Z&=\tan^{-1}\frac{\rho_1-a}{\sqrt{1-a^2}}+\tan^{-1}\frac{\rho_2-a}{\sqrt{1-a^2}},
\end{aligned}\right\}\quad(5)$$

which give rise to the following equations, ρ_1 and ρ_2 being eliminated:

$$\begin{aligned}A\tan X\tan Y\tan Z+B\tan X\tan Y+C\tan X\tan Z+D\tan Y\tan Z\\
+E\tan X+F\tan Y+G\tan Z=0,\end{aligned}\qquad(6)$$

in which the constants $A, B, \ldots, G$ have the following values:

$$
\begin{aligned}
A &= (\alpha_1 - \alpha_2)\left[\alpha_1\alpha_2 - a(\alpha_1 + \alpha_2) + 2a^2 - 1\right] + \alpha_1 k_2^2 - \alpha_2 k_1^2 - a(k_2^2 - k_1^2),\\
B &= \sqrt{1-a^2}\,(\alpha_1^2 - \alpha_2^2 + 2a\alpha_2 - 2a\alpha_1 + k_2^2 - k_1^2),\\
C &= k_2(k_1^2 - \alpha_1^2 + 2\alpha_1\alpha_2 - 2a\alpha_2 + 2a^2 - 1),\\
D &= k_1(\alpha_2^2 - k_2^2 + 2a\alpha_1 - 2\alpha_1\alpha_2 + 1 - 2a^2),\\
E &= 2\sqrt{1-a^2}\,k_2(\alpha_2 - a),\\
F &= 2\sqrt{1-a^2}\,k_1(a - \alpha_1),\\
G &= 2k_1k_2(\alpha_1 - \alpha_2).
\end{aligned}
$$

It remains now to transform the origin to the center of symmetry and to find the coordinates of this center. If we start with equations (6), putting $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$, and express the conditions that the resulting equation shall reduce to the form

$$A' \tan X \tan Y + B' \tan X \tan Z + C' \tan Y \tan Z + D' = 0,$$

we obtain a set of equations involving $\tan\xi$, $\tan\eta$, $\tan\zeta$ which appear somewhat difficult to solve by ordinary methods. To avoid this difficulty we substitute for the trigonometric functions their exponential values, so that we obtain the following equation of the surface:

$$
\left.
\begin{aligned}
&(-B - C - D + Ai - Ei - Fi - Gi)\, e^{i(X+Y+Z)}\\
&\qquad + (-B - C - D - Ai + Ei + Fi + Gi)\, e^{-i(X+Y+Z)}\\
&+ (C + D - B - Ai - Fi + Gi)\, e^{i(X+Y-Z)}\\
&\qquad + (C + D - B + Ai + Ei + Fi - Gi)\, e^{-i(X+Y-Z)}\\
&+ (B + D - C + Ai - Ei + Fi - Gi)\, e^{i(X+Z-Y)}\\
&\qquad + (B + D - C - Ai + Ei - Fi + Gi)\, e^{-i(X+Z-Y)}\\
&+ (B + C - D - Ai + Ei - Fi - Gi)\, e^{i(Y+Z-X)}\\
&\qquad + (B + C - D + Ai - Ei + Fi + Gi)\, e^{-i(Y+Z-X)} = 0,
\end{aligned}
\right\} \quad (6')
$$

which may be written in the form

$$
\begin{aligned}
A_1 + B_1 e^{2iX} + C_1 e^{2iY} + D_1 e^{2iZ} + E_1 e^{2i(X+Z)} + F_1 e^{2i(X+Y)} + G_1 e^{2i(Y+Z)}&\\
+ H_1 e^{2i(X+Y+Z)} = 0.& \qquad (6'')
\end{aligned}
$$

Putting now in (6') $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$ and expressing the condition of symmetry, viz.:

$$A_1 = H_1,\quad B_1 = G_1,\quad C_1 = E_1,\quad D_1 = F_1,$$

we obtain the following equations:

$$\begin{aligned} \xi+\eta+\zeta &= \tan^{-1}\frac{B+C+D}{G+F+E-A}, \\ \xi+\eta-\zeta &= \tan^{-1}\frac{C+D-B}{G-A-E-F}, \\ \xi+\zeta-\eta &= \tan^{-1}\frac{B+D-C}{E+G-A-F}, \\ \eta+\zeta-\xi &= \tan^{-1}\frac{B+C-D}{A+G-E-F}. \end{aligned} \tag{7}$$

Solving these equations, we have, using all four equations (7),

$$\begin{aligned} \xi &= \frac{1}{2}\left[\tan^{-1}\frac{C+D-B}{G-A-E-F}+\tan^{-1}\frac{B+D-C}{E+G-A-F}\right], \\ \eta &= \frac{1}{2}\left[\tan^{-1}\frac{C+D-B}{G-A-E-F}+\tan^{-1}\frac{B+C-D}{A-E-F+G}\right], \\ \zeta &= \frac{1}{2}\left[\tan^{-1}\frac{B+D-C}{E+G-A-F}+\tan^{-1}\frac{B+C-D}{A-E-F+G}\right]. \end{aligned} \tag{8}$$

Since, moreover, the relation

$$\frac{E_1 G_1}{C_1 B_1}=\frac{H_1 D_1}{A_1 F_1} \tag{9}$$

must necessarily be satisfied, if the surface is to be symmetrical, the equations (7) are all satisfied, so that (9) may be replaced by the equivalent one,

$$\begin{aligned} \tan^{-1}\frac{B+C+D}{E+F+G-A} = \tan^{-1}\frac{C+D-B}{G-A-E-F} &+ \tan^{-1}\frac{B+D-C}{E-F+G-H} \\ &+ \tan^{-1}\frac{B+C-D}{A-E-F+G}. \end{aligned}$$

We shall not verify this relation, as it would involve long and tedious algebraic calculations; it is moreover unnecessary, its truth being known *a priori.*

The surface (6″) now takes the form

$$\begin{aligned} A_1(1+e^{2i(X+Y+Z)})+B_1(e^{2i(Y+Z)}+e^{2iX})+C_1(e^{2i(X+Z)}+e^{2iY}) \\ +D_1(e^{2i(Y+Z)}+e^{2iX})=0, \end{aligned} \tag{10}$$

which may easily be reduced back to the form

$$A'\tan X\tan Y+B'\tan X\tan Z+C'\tan Y\tan Z+D'=0, \tag{11}$$

where the coefficients A', ..., D' are found from the equations

$$\begin{aligned} A'+B'+C'+D' &= \sqrt{(B+C+D)^2+(E+F+G-A)^2}, \\ A'-B'-C'+D' &= \sqrt{(C+D-B)^2+(G-F-E-A)^2}, \\ -A'+B'-C'+D' &= \sqrt{(B+D-C)^2+(A+F-E-G)^2}, \\ -A'-B'+C'+D' &= \sqrt{(B+C-D)^2+(E-F-G-A)^2}. \end{aligned}$$

We have not carried out these calculations in detail, as they do not present any serious difficulties. We have then the

THEOREM: *To a unicursal quartic with three conjugate points there corresponds a translation-surface of the form*

$$A'\tan X\tan Y+B'\tan X\tan Z+C'\tan Y\tan Z+D'=0. \tag{11}$$

If we transform (10) by means of the transformation $X'=2iX$, $Y'=2iY$, $Z'=2iZ$, it takes the same form as was obtained in the case where the double points of the quartic have real and distinct tangents [see p. 177, (9)], viz.:

$$A_1(1+e^{X'+Y'+Z'})+B_1(e^{Y+Z}+e^X)+C_1(e^{X+Z}+e^Y)+D_1(e^{X+Y}+e^Z)=0. \tag{9}$$

Now since a transformation of the form

$$X'=2iX, \qquad Y'=2iY, \qquad Z'=2iZ$$

leaves the quartic in the plane at infinity unaltered, we may collect the result obtained in the following form:

THEOREM: *To a unicursal quartic having non-consecutive double points with distinct tangents, these tangents being either both real, or both imaginary, in pairs, there corresponds a translation-surface which may be thrown into either of the following forms:*

$$A(1+e^{X+Y+Z})+B(e^{Y+Z}+e^X)+C(e^{X+Z}+e^Y)+D(e^{X+Y}+e^Z)=0, \tag{10}$$

$$A'\tan X\tan Y+B'\tan X\tan Z+C'\tan Y\tan Z+D'=0. \tag{11}$$

If we put $X=X'+\pi i$, $Y=Y'+\pi i$, $Z=Z'+\pi i$ in (10) and $X=X'+\pi$, $Y=Y'+\pi$, $Z=Z'+\pi$ in (11), these equations may also be written:

$$A(1-e^{X'+Y'+Z'})+B(e^{Y'+Z'}-e^{X'})+C(e^{X'+Z'}-e^{Y'})+D(e^{X'+Y'}-e^{Z'})=0, \tag{10'}$$

$$A'\tan Z'+B'\tan Y'+C'\tan X'+D'\tan X'\tan Y'\tan Z'=0, \tag{11'}$$

which are sometimes more convenient, inasmuch as *the center of symmetry is here situated on the surface.*

III.

Quartics Having Two Double Points with Real Tangents and One Conjugate Point.

Let the conjugate point be at $x=0,\ y=\infty$. We have now to integrate equations (3′), p. 174, on the hypothesis, $a<1,\ b>1,\ c>1$, and after a suitable *real* transformation, in order to avoid extraneous factors, we have

$$\begin{aligned} X &= \tan^{-1}\frac{\rho_1+b}{\sqrt{1-b^2}}+\tan^{-1}\frac{\rho_2+b}{\sqrt{1-b^2}}, \\ 2Y &= \log\frac{(\rho_1-\alpha_2)\,(\rho_2-\alpha_2)}{(\rho_1-\beta_2)\,(\rho_2-\beta_2)}, \\ 2Z &= \log\,(\rho_1-k_1)\,(\rho_2-k_1), \end{aligned}$$

where α_2, β_2, as before, are the roots of the equation

$$\rho^2+(4b+2mc)\,\rho+m^2+4b^2+4bmc=0, \text{ and } k_1=-\frac{2\,(b+mc)}{1-m^2}.$$

Eliminating ρ_1 and ρ_2 we have the equation

$$\begin{vmatrix} 1 & k_1 & k_1^2-e^{2Z} \\ 1-e^{2Y} & \alpha_2-\beta_2\,e^{2Y} & \alpha_2^2-\beta_2^2\,e^{2Y} \\ -\tan X & b\tan X+\sqrt{1-b^2} & (1-2b^2)\tan X-2b\sqrt{1-b^2} \end{vmatrix}=0,$$

or, developed,

$$\tan X=\frac{A\,e^{2(Y+Z)}+B\,e^{2Y}+C\,e^{2Z}+D}{A'\,e^{2(Y+Z)}+B'\,e^{2Y}+C'\,e^{2Z}+D'}, \tag{1}$$

where

$$\begin{aligned} &A=-\sqrt{1-b^2}, && A'=b+\beta_2, \\ &B=\sqrt{1-b^2}(k_1-\beta_2)(k_1+\beta_2+2b), && B'=(k_1-\beta_2)(1-2b^2-k_1\beta_2-k_1b-\beta_2b), \\ &C=\sqrt{1-b^2}, && C'=-b-\alpha_2, \\ &D=\sqrt{1-b^2}(\alpha_2-k_1)(\alpha_2+k_1+2b), && D'=(\alpha_2-k_1)(1-2b^2-\alpha_2k_1-\alpha_2b-k_1b). \end{aligned} \tag{2}$$

The equation (1) may be simplified just as in the former case by transforming to the center of symmetry. Putting $X=X'+\xi,\ Y=Y'+\eta,\ Z=Z'+\zeta$, and expressing the condition of symmetry, we have

$$\begin{aligned} (A-A'\tan\xi)\,e^{2(\eta+\zeta)} &= -(D-D'\tan\xi), \\ (B-B'\tan\xi)\,e^{2(\eta-\zeta)} &= -(C-C'\tan\xi), \\ (A'+A\tan\xi)\,e^{2(\eta+\zeta)} &= \quad D'+D\tan\xi, \\ (B'+B\tan\xi)\,e^{2(\eta-\zeta)} &= \quad C'+C\tan\xi. \end{aligned} \tag{3}$$

From these equations we find that $\tan\xi$ must be a common root of the following two quadratic equations:

$$\left.\begin{array}{l}\text{(a)}\ (AD'+DA')\tan^2\xi+2\,(A'D'-DA)\tan\xi-(AD'+DA')=0,\\ \text{(b)}\ (BC'+CB')\tan^2\xi+2\,(B'C'-BC)\tan\xi-(BC'+CB')=0.\end{array}\right\} \tag{4}$$

The condition that these equations shall have a common root is

$$(AD'+DA)\,(B'C'-BC)=(BC'+B'C)\,(A'D'-DA), \tag{5}$$

which is seen to be identically satisfied by the values of $A, \ldots, D, A', \ldots, D'$ obtained from (2). Calling the roots of (4) α and $-\frac{1}{\alpha}$, we have, by solving,

$$\xi=\tan^{-1}\alpha, \qquad \xi=\tan^{-1}\alpha-\frac{\pi}{2},$$

of which either value may be taken without influencing the form of (1) as to symmetry. Solving (3) we have

$$\eta=\tfrac{1}{2}\log\frac{(D'\tan\xi-D)\,(C'\tan\xi-C)}{(A-A'\tan\xi)\,(B-B'\tan\xi)},$$
$$\zeta=\tfrac{1}{2}\log\frac{(D'\tan\xi-D)\,(B-B'\tan\xi)}{(A-A'\tan\xi)\,(C'\tan\xi-C)}.$$

The surface now reduces to the form

$$\tan X=\frac{A_1\,(e^{2(Y+Z)}-1)+B_1\,(e^{2Y}-e^{2Z})}{A_1'\,(e^{2(Y+Z)}+1)+B_1'\,(e^{2Y}+e^{2Z})}, \tag{6}$$

in which $A_1=D-D'\tan\xi$, $B_1=C-C'\tan\xi$, $A_1'=D'+D\tan\xi$, and $B_1'=C'+C\tan\xi$. We have then the

THEOREM: *To a quartic having two double points with distinct and real tangents and one conjugate point there corresponds a translation-surface of the form*

$$\tan X=\frac{Ae^{2(Y+Z)}+Be^{2Y}+Ce^{2Z}+D}{A'e^{2(Y+Z)}+B'e^{2Y}+C'e^{2Z}+D'}, \tag{7}$$

with the following identical relation between the coefficients:

$$(AD'+DA')\,(B'C'-BC)=(BC'+CB')\,(A'D'-DA). \tag{5}$$

The surface (6) has two imaginary and one real period. By using the transformation $X=iX'$, $Y=iY'$, $Z=iZ'$, which does not affect the quartic curve, we may transform it into a surface having two real and one imaginary period. We have

$$\frac{e^{-X}-e^{X}}{e^{-X}+e^{X}}=\frac{iA_1\,(e^{2i(Y+Z)}-1)+iB_1\,(e^{2iY}-e^{2iZ})}{A_1'\,(e^{2i(Y+Z)}+1)+B_1'\,(e^{2iY}+e^{2iZ})},$$

which may be written

$$\begin{aligned} &e^{-X}\left[(A_1'-iA_1)\,e^{2i(Y+Z)}+(B_1'-iB_1)\,e^{2iY}+(B_1'+iB_1)\,e^{2iZ}+A_1'+iA_1\right] \\ &-e^{X}\left[(A_1'+iA_1)\,e^{2i(Y+Z)}+(B_1'+iB_1)\,e^{2iY}+(B_1'-iB_1)\,e^{2iZ}+A_1'-iA_1\right]=0. \end{aligned} \tag{8}$$

By principle of symmetry this equation may also be written, putting $X=-X$, $Y=-Y$, $Z=-Z$,

$$\begin{aligned} &e^{X}\left[(A_1'-iA_1)\,e^{-2i(Y+Z)}+(B_1'-iB_1)\,e^{-2iY}+(B_1'+iB_1)\,e^{-2iZ}+A_1'+iA\right] \\ &-e^{-X}\left[(A_1'+iA_1)e^{-2i(Y+Z)}+(B_1'+iB_1)e^{-2iY}+(B_1'-iB_1)e^{-2iZ}+A_1'-iA\right]=0. \end{aligned} \tag{9}$$

Adding (8) and (9) and introducing the trigonometric equivalents, we have

$$e^{2X}=-\frac{L\tan Y\tan Z+M\tan Y+N\tan Z+P}{L\tan Y\tan Z-M\tan Y-N\tan Z+P}, \tag{10}$$

where $L=\dfrac{B_1'-A_1'}{2}$, $P=\dfrac{B_1'+A_1'}{2}$, $M=\dfrac{A_1+B_1}{2}$, $N=\dfrac{A_1-B_1}{2}$.

IV.

Quartics Having One Double Point with Distinct Tangents and Two Conjugate Points.

We have in this case $b<1$, $c<1$, $a>1$, the conjugate points being $x=0$, $y=\infty$; $x=\infty$, $y=0$. On this hypothesis, integrating equations (3′), p. 174, we have

$$\begin{aligned} X&=\tan^{-1}\frac{\rho_1+b}{\sqrt{1-b^2}}+\tan^{-1}\frac{\rho_2+b}{\sqrt{1-b^2}}, \\ Y&=\tan^{-1}\frac{\rho_1+2b+mc}{m\sqrt{1-c^2}}+\tan^{-1}\frac{\rho_2+2b+mc}{m\sqrt{1-c^2}}, \\ Z&=\log\,(\rho_1-k_1)\,(\rho_2-k_1). \end{aligned} \tag{1}$$

Eliminating we have

$$e^{2Z}=\frac{A\tan X\tan Y+B\tan X+C\tan Y+D}{A'\tan X\tan Y+B'\tan X+C'\tan Y}, \tag{1'}$$

where $A, \ldots, D, A', \ldots, C'$ have the following values:

$$\begin{aligned} A&=(2b+mc)\left[1+k_1^2+bmc+2bk_1+k_1mc\right]+k_1(1-2b^2) \\ &\qquad\qquad +k_1^2b+m^2(1-c^2)(b_1+k_1), \\ B&=m\sqrt{1-c^2}\left[1+k_1^2+2b^2+2bmc+4k_1b+2k_1mc\right], \\ C&=\sqrt{1-b^2}\,(2m^2c^2-m^2+2bmc-k_1^2-2k_1b), \\ D&=2m\sqrt{1-b^2}\sqrt{1-c^2}\,(b+mc), \\ A'&=b+mc,\quad B'=m\sqrt{1-c^2},\quad C'=-\sqrt{1-b^2}. \end{aligned} \tag{2}$$

Transforming the center of symmetry, we have

$$e^{2Z} = \frac{E \tan X \tan Y + F \tan X + G \tan Y + H}{E \tan X \tan Y - F \tan X - G \tan Y + H}, \tag{3}$$

where the coefficients have the following values:

$$\begin{aligned}
E &= A - B \tan\eta - C \tan\xi + D \tan\xi \tan\eta = e^{\zeta}(A' - B' \tan\eta - C' \tan\xi),\\
F &= A \tan\eta + B - C \tan\xi \tan\eta - D \tan\xi = -e^{\zeta}(A' \tan\eta + B' - C' \tan\xi \tan\eta),\\
G &= A \tan\xi - B \tan\eta \tan\xi + C - D \tan\eta = -e^{\zeta}(A' \tan\xi - B' \tan\xi \tan\eta + C'),\\
H &= A \tan\xi \tan\eta + B \tan\xi + C \tan\eta + D = e^{\zeta}(A' \tan\xi \tan\eta + B' \tan\xi + C' \tan\eta).
\end{aligned} \tag{4}$$

From these equations we obtain by elimination of e^{ζ}

$$\begin{aligned}
[(A-D)(B'+C') + A'(B+C)] \tan^2(\eta+\xi) + 2A'(A-D)\tan(\eta+\xi)\\
-(A-D)(B'+C') - A'(B+C) = 0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
[(A+D)(C'-B') + A'(C-D)] \tan^2(\eta-\xi) + 2A'(A+D)\tan(\eta-\xi)\\
-(A+D)(B'-C') - A'(B-C) = 0,
\end{aligned} \tag{6}$$

$$\begin{aligned}
2A'D \tan(\eta+\xi)\tan(\eta-\xi) + [(B'+C')(A+D) - (B+C)A'] \tan(\eta-\xi)\\
+ [A'(B-C) - (B'-C')(A-D)] \tan(\eta+\xi) + 2(BC' - CB') = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
2(CB' - BC') \tan(\eta+\xi)\tan(\eta-\xi) + [(A-D)(C'-B') - (C-B)A'] \tan(\eta-\xi)\\
+ [(B'+C')(A+D) - (B+C)A'] \tan(\eta+\xi) - 2A'D = 0.
\end{aligned} \tag{8}$$

From (7) and (8) we easily find

$$\eta = \tfrac{1}{2} \tan^{-1} \frac{CB' - BC' + A'D}{C'A + BD' - A'C'},$$

$$\xi = \tfrac{1}{2} \tan^{-1} \frac{BC' + A'D - CB'}{B'A + C'D - BA'}.$$

Calling one of the two reciprocal roots of (5) α, we have

$$\zeta = \log \frac{A - D - (B+C)\alpha}{A' - (B'+C')\alpha},$$

which three coordinates will satisfy all four equations provided the following relation exists:

$$\frac{TU + RS}{(A-D)(B'+C') + A'(B+C)} = \frac{S^2 + U^2}{2A'(D-A)},$$

where

$$R = 2A'D, \quad S = (B'+C')(A+D) - A'(B+C),$$
$$T = A'(B+C) - (B'-C')(A-D), \quad U = 2(BC' - B'C).$$

If we substitute the values $A, \ldots, D, A', \ldots, C'$ from (2) in this relation it is seen to be satisfied identically.*

As in the former case, we may now prove that by means of the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$ we may put (1') in the form

$$\tan X' = \frac{E'(e^{2(X+Y)} - 1) + F'(e^{2X} - e^{2Y})}{E_1'(e^{2(X+Y)} + 1) + F_1'(e^{2X} + e^{2Y})},$$

so that combining the results of III and IV we have the following

THEOREM: *To a unicursal quartic with two double points having distinct tangents and one conjugate point, or two conjugate points and one double point, there correspond* ∞^3 *types of translation-surfaces that can be generated in four different ways. The general equation of these surfaces may be put into either of the two forms:*

$$\text{(a)} \qquad \tan X = \frac{A_1(e^{2(Y+Z)} - 1) + B_1(e^{2Y} - e^{2Z})}{A_1'(e^{2(Y+Z)} + 1) + B_1'(e^{2Y} + e^{2Z})},$$

$$\text{(b)} \qquad e^{2X} = \frac{E \tan Y \tan Z + F \tan Y + G \tan Z + H}{E \tan Y \tan Z - F \tan Y - G \tan Z + H}.$$

The form (a) *is transformed into* (b) *by means of the transformation* $X = iX'$, $Y = iY'$, $Z = iZ'$.

V.

Quartics with One Cusp and Two Double Points.

1. Let the double points have real tangents. Putting $a = 1$ in (3'), p. 174, and remembering that b and c are both greater than unity, we have

$$X = \log \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{(\rho_1 - \beta_1)(\rho_2 - \beta_1)},$$
$$Y = \log \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)},$$
$$Z = \rho_1 + \rho_2.$$

Eliminating ρ_1 and ρ_2, we obtain the surface

$$Z = \frac{(\beta_1^2 - \beta_2^2)e^{X+Y} + (\alpha_2^2 - \beta_1^2)e^X + (\beta_2^2 - \alpha_1^2)e^Y + \alpha_1^2 - \alpha_2^2}{(\beta_2 - \beta_1)e^{X+Y} + (\beta_1 - \alpha_2)e^X + (\alpha_1 - \beta_2)e^Y + \alpha_2 - \alpha_1},$$

and transforming to the center ξ, η, ζ, we find

$$Z = \frac{A(e^{X+Y} - 1) + B(e^X - e^Y)}{A'(e^{X+Y} + 1) + B'(e^X + e^Y)}, \tag{1}$$

* The details of the algebraic work have been omitted as unnecessary.

where

$$A = [\alpha_2^2 - \alpha_1^2 + \zeta(\alpha_2 - \alpha_1)], \quad B = [\alpha_1^2 - \beta_2^2 + \zeta(\alpha_1 - \beta_2)]\, e^{\eta},$$
$$A' = \alpha_2 - \alpha_1, \qquad B' = (\alpha_1 - \beta_2)\, e^{\eta},$$

the coordinates of the center of symmetry being

$$\xi = \tfrac{1}{2} \log \frac{(\alpha_2 - \alpha_1)(\alpha_1 - \beta_2)}{(\beta_2 - \beta_1)(\beta_1 - \alpha_2)}, \quad \eta = \tfrac{1}{2} \log \frac{(\alpha_2 - \alpha_1)(\beta_1 - \alpha_2)}{(\alpha_1 - \beta_2)(\beta_2 - \beta_1)},$$
$$\zeta = -\frac{\alpha_1 + \alpha_2 + \beta_1 + \beta_2}{2}.$$

2. When the double points are conjugate points, that is $b < 1$, $c < 1$, the surface, when transformed to its center of symmetry as origin, takes the form

$$Z = \frac{A \tan X + B \tan Y}{A' \tan X \tan Y + B'}, \tag{2}$$

which may be derived from equations (3′), p. 174, by putting $m = 1$. Hence the

THEOREM: *To a unicursal quartic with one cusp, and two double points whose tangents may be either real or imaginary, there correspond* ∞^2 *types of translation-surfaces that can be generated in four different ways. The equation of these surfaces may be thrown into either of the two following forms* (corresponding to real and imaginary pairs of tangents):

$$Z = \frac{A(e^{X+Y} - 1) + B(e^X - e^Y)}{A'(e^{X+Y} + 1) + B'(e^X + e^Y)}, \tag{1}$$
$$Z = \frac{A \tan X + B \tan Y}{A' \tan X \tan Y + B'}. \tag{2}$$

3. If only one of the double points is a conjugate point, we have, since now $a = 1$, $b < 1$, $c > 1$ (p. 174, (3′)),

$$X = \tan^{-1} \frac{\rho_1 + b}{\sqrt{1 - b^2}} + \tan^{-1} \frac{\rho_2 + b}{\sqrt{1 - b^2}},$$
$$Y = \log \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)},$$
$$Z = \rho_1 + \rho_2,$$

which gives rise to the following equations, eliminating ρ_1 and ρ_2:

$$Z = \frac{(2b^2 - 1 - \beta_2^2) \tan X . e^Y + (\alpha_2^2 + 1 - 2b^2) \tan X + 2b\sqrt{1 - b^2}\, e^Y - 2b\sqrt{1 - b^2}}{-(b + \beta_2) \tan X . e^Y + (b + \alpha_2) \tan X - \sqrt{1 - b^2}\, e^Y + \sqrt{1 - b^2}}$$

which by transformation to the center of symmetry takes the form

$$Z = \frac{A \tan X . (e^Y + 1) + B(e^Y - 1)}{A' \tan X . (e^Y - 1) + B'(e^Y + 1)}, \tag{3}$$

so that we have the

THEOREM: *To a quartic having one cusp, one double point with real tangents and one conjugate point correspond* ∞^2 *translation-surfaces of the form*

$$Z = \frac{A \tan X . (e^Y + 1) + B(e^Y - 1)}{A' \tan X . (e^Y - 1) + B'(e^Y + 1)}. \tag{3}$$

It will be noticed that in this case the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$ leaves the surface in the same form as before.

VI.

Quartics with a Double Point and Two Cusps.

1. The double point has a pair of real tangents. In this case we have $b = c = 1$ and $a > 1$. Equations (3′), p. 174, give us by integrating:

$$\begin{aligned} X &= \frac{1}{\rho_1 + 1} + \frac{1}{\rho_2 + 1}, \\ Y &= \frac{1}{\rho_1 + 2 + m} + \frac{1}{\rho_2 + 2 + m}, \\ Z &= \log\left(\rho_1 + \frac{2}{1 - m}\right)\left(\rho_2 + \frac{2}{1 - m}\right), \end{aligned} \tag{1}$$

from which we obtain the surface

$$[X - Y - (1+m)XY]e^Z - \frac{(m+1)^2(2m-1)}{(1-m)^2}X + \frac{m(m+1)^2(2-m)}{(1-m)^2}Y$$
$$+ \frac{m(1+m)^3}{(1-m)^2}XY + 2(1+m) = 0,$$

which may be written, putting $(1-m)^2 e^Z = e^{Z'}$,

$$e^{Z'} = \frac{(m+1)^2(2m-1)X + m(m+1)^2(m-2)e^Y - m(1+m)^3 XY - 2(1+m)(1-m)^2}{X - Y - (1+m)XY}.$$

We now put $Z' = Z - \log k_3$, $X = X' + k_1$, $Y = Y' + k_2$, k_3 being a positive quantity, and express the condition that the coefficients of X and Y in the numerator shall equal the coefficients of X and Y in the denominator taken with opposite signs, while the absolute term and the coefficient of XY in

the numerator shall equal the corresponding terms in the denominator. We thus obtain the following four equations:

$$k_1 k_3 (1+m)^2 (2m-1) + k_2 k_3 m (1+m)^2 (m-2) - k_1 k_2 k_3 m (1+m)^3 - 2k_3 (1+m)(1-m)^2 = k_1 - k_2 - k_1 k_2 (1+m),$$

$$k_3 m (1+m)^3 = 1+m,$$

$$k_3 (1+m)^2 (2m-1) - k_2 k_3 m (1+m)^3 = -1 + k_2 (1+m),$$

$$k_3 m (1+m)^2 (m-2) - k_1 k_3 m (1+m)^3 = 1 + k_1 (1+m).$$

Solving the three last equations, we obtain

$$k_1 = \frac{m-3}{2(1+m)}, \quad k_2 = \frac{3m-1}{2m(1+m)}, \quad k_3 = \frac{1}{m(1+m)^2},$$

which, substituted in the first, reduces it to an identity. The equation of the surface is:

$$e^Z = \frac{2(1-m)^2 - \frac{m-1}{2m} X - \frac{m-1}{2} Y + (1+m) XY}{2(1-m)^2 + \frac{m-1}{2} X + \frac{m-1}{2} Y + (1+m) XY},$$

or, putting $\frac{m-1}{2m} X$ equal to a new X and $\frac{m-1}{2} Y$ equal to a new Y,

$$e^Z = \frac{2(1-m)^2 - X - Y + \frac{4m(1+m)}{(1-m)^2} XY}{2(1-m)^2 + X + Y + \frac{4m(1+m)}{(1-m)^2} XY}.$$

If now we make use of the transformation $X = X' + iY'$, $Y = X' - iY'$, which amounts to transforming the quartic into a limaçon with a conjugate point, we obtain a surface which has a striking resemblance to the Cardioid surface obtained in my previous paper.* The equation of this surface is:

$$e^Z = \frac{2(1-m)^2 - 2X + \frac{4m(1+m)}{(1-m)^2}(X^2+Y^2)}{2(1-m)^2 + 2X + \frac{4m(1+m)}{(1-m)^2}(X^2+Y^2)}. \tag{3}$$

Every section parallel to the XY-plane is a circle which for $Z=0$ becomes the Y-axis. If we put $e^Z = k$, $A = 2(1-m)^2$, $B = \frac{4m(1+m)}{(1-m)^2}$, this section may be written

$$B(k-1)(X^2+Y^2) + 2(k+1)X + A(k-1) = 0,$$

* *American Journal of Mathematics*, vol. 29, p. 378. (See plate of model.)

which shows that *if the circular section be real, we must have* $AB < \left(\frac{k+1}{k-1}\right)^2$, *so that to any given type, that is, for any given value of m, the surface will have two umbilical points on the Z-axis at equal distances above and below the origin.* This point will be at infinity when $8m(1+m) = 1$, or $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3}{2}}$. *If the product AB is* < 1, *there will be no real umbilical point. If AB is* > 1, *this point is determined by the equation*

$$\left(\frac{e^Z+1}{e^Z-1}\right)^2 = AB = 8m(1+m),$$

which, regarded as an equation determining Z, has two real and finite roots.

Examples: 1. $m = \frac{-1}{2}$. The surface extends to infinity in both directions along the Z-axis and has no umbilical points (Fig. 1).

2. $m = -2$. The surface has two umbilical points at $Z = \pm \log \frac{5}{3}$ (Fig. 2). In both cases the projection of the surface on the XZ-plane has been given.

The surface (3) has an imaginary period, but if we transform to the imaginary space (iX, iY, iZ), we obtain one having a real period. Writing the surface in the form

$$e^Z = \frac{A - 2X + B(X^2+Y^2)}{A + 2X + B(X^2+Y^2)}$$

and using the transformation, we have

$$\frac{e^{2iZ}-1}{e^{2iZ}+1} = \frac{1}{i}\tan Z = \frac{-4Xi}{A - 4B(X^2+Y^2)},$$

or,

$$\tan Z = \frac{4X}{A - 4B(X^2+Y^2)}, \tag{4}$$

one branch of which is contained entirely in a space between the parallel planes $Z = -\pi$ and $Z = \pi$. Any section $\tan Z =$ const. is the circle

$$X^2 + Y^2 + \frac{X}{Bk} = \frac{A}{4B},$$

or,

$$\left(X + \frac{1}{2Bk}\right)^2 + Y^2 = \frac{1}{4B^2}\left(AB + \frac{1}{k^2}\right),$$

from which it is obvious that, whenever AB is positive, the surface will extend to infinity along the Z-axis. If, however, AB is negative, the surface will

become imaginary somewhere between $Z=0$ and $Z=\pm\frac{\pi}{2}$, so that there will be an umbilical point at $k=\pm\frac{1}{\sqrt{-AB}}$.

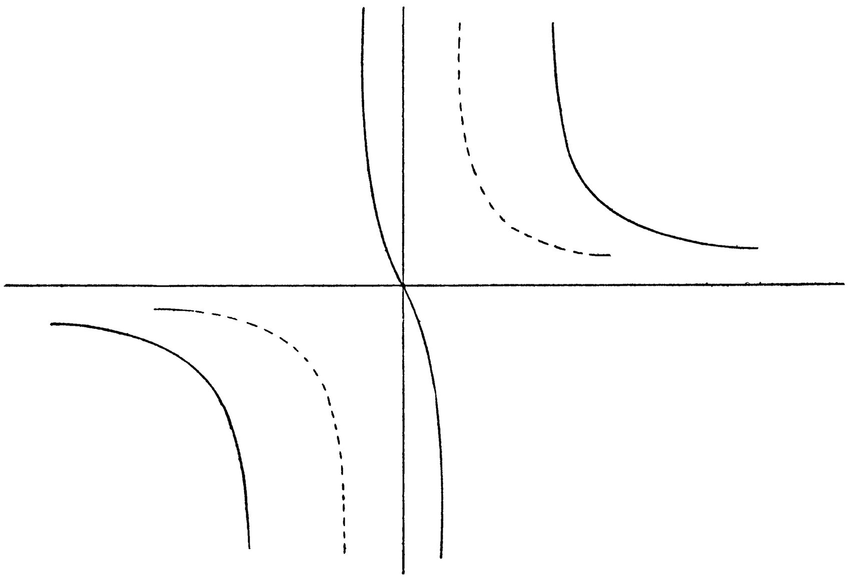

FIG. 1.

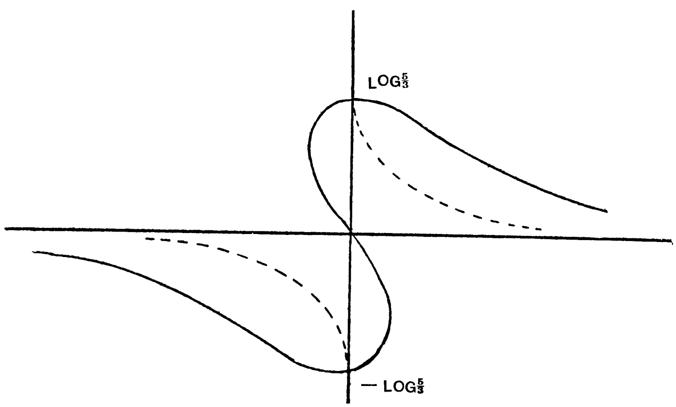

FIG. 2.

Example 1. If $A=4B$, m is equal to 0.07 nearly; for $Z=\frac{\pi}{2}$ we obtain the unit circle $X^2+Y^2=1$ (Fig. 3).

Example 2. Let $AB = -1$, $m = \dfrac{-1 \pm \sqrt{\frac{1}{2}}}{2}$. The umbilical point is $Z = \pm \dfrac{\pi}{4}$ (Fig. 4). In both examples the locus of the centers of the circular sections has been indicated by the dotted curve.

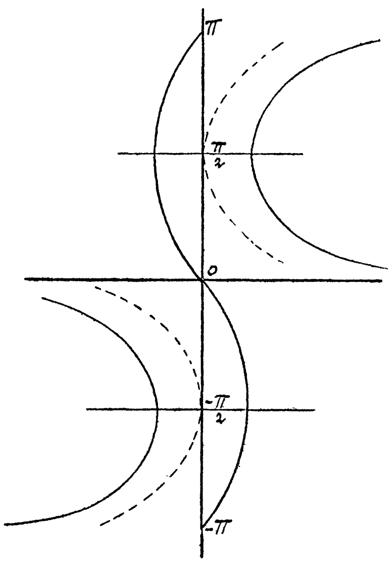

FIG. 3.

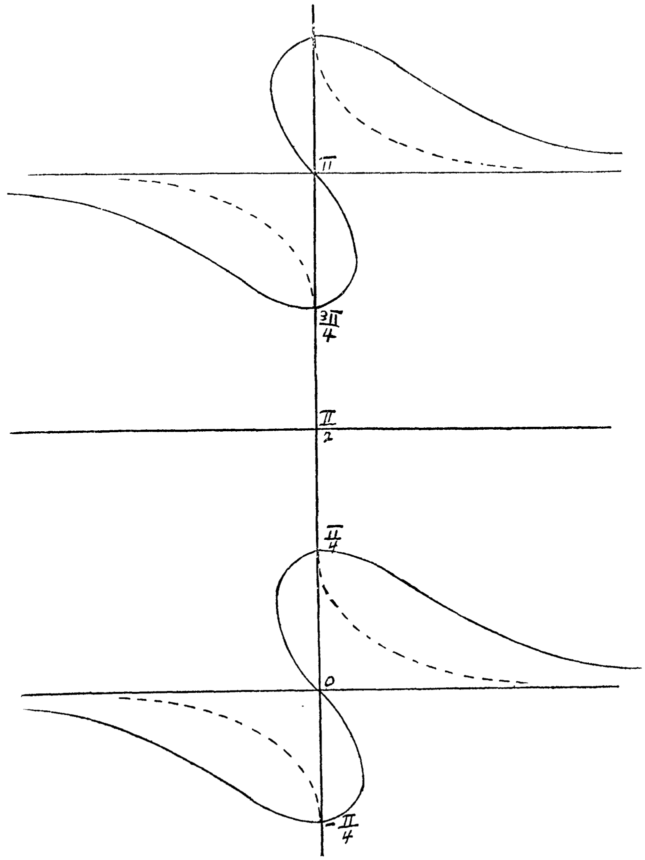

FIG. 4.

VII.

Quartics Having Two Cusps and a Conjugate Point.

In this case we put $m=1$, $a=1$, $c=1$, in (3′), p. 174, while b is less than unity. Integrating we have, omitting extraneous factors,

$$X = \tan^{-1}\frac{\rho_1 + b}{\sqrt{1-b^2}} + \tan^{-1}\frac{\rho_2 + b}{\sqrt{1-b^2}},$$

$$Y = \frac{1}{\rho_1 + 1 + 2b} + \frac{1}{\rho_2 + 1 + 2b},$$

$$Z = \rho_1 + \rho_2,$$

from which by elimination we obtain

$$\tan X = \frac{(Z + 2b)\,Y\sqrt{1-b^2}}{2(1+b)^2\,Y + (b+1)\,YZ - Z - 2 - 4b};$$

transforming, putting $Z + 2b = Z'$, this may be written:

$$\tan X = \frac{\sqrt{1-b^2}\,YZ}{2(1+b)\,Y + (1+b)\,YZ - Z - 2b - 2}.$$

The center of symmetry may now be found just as before. We have

$$\frac{\tan X + k_1}{1 - k_1 \tan X} = \frac{(Y+k_2)(Z+k_3)(1 - k_1\tan X)\sqrt{1-b^2}}{2(1+b)(Y+k_2) + (1+b)(Y+k_2)(Z+k_3) - Z - k_3 - 2b - 2},$$

in which the terms in $YZ \tan X$, $\tan X$, Y, Z must vanish. We have therefore the four equations:

$$\begin{aligned}
&-\sqrt{1-b^2}\,k_1 - (1+b) = 0,\\
&\sqrt{1-b^2}\,k_3 - 2k_1(1+b) - k_1k_3(1+b) = 0,\\
&\sqrt{1-b^2}\,k_2 - k_1k_2(1+b) + k_1 = 0,\\
&-\sqrt{1-b^2}\,k_1k_2k_3 + k_3 + 2b + 2 - 2(1+b)\,k_2 - (1-b)\,k_2k_3 = 0.
\end{aligned}$$

Solving, we find

$$k_1 = -\frac{1+b}{\sqrt{1-b^2}}, \quad k_2 = \tfrac{1}{2}, \quad k_3 = -(1+b),$$

which values satisfy the fourth equation. The equation now reduces to the form

$$\tan X = \frac{(1+b)(1+2b) - 2(1+b)\,YZ}{\sqrt{1-b^2}\,(Z - 2(1+b)\,Y)},$$

which may be simplified by putting $\dfrac{\sqrt{1-b^2}}{1+b}\,Z = Z'$, $-2\sqrt{1-b^2}\,Y = Y'$, so that we have

$$\tan X = \frac{(1+2b) + \dfrac{1}{1-b}\,ZY}{Z+Y}.$$

Every section $X = \text{const.}$ is a rectangular hyperbola; one branch of the surface is contained entirely in the space between the planes $X = \frac{\pi}{2}$, $X = -\frac{\pi}{2}$. If we transform the surface, using the transformation $Z = Z' + iY'$, $Y = Z' - iY'$, we have

$$\tan X = \frac{1 + 2b + \dfrac{1}{1-b}(Z^2 + Y^2)}{2Z}, \tag{5}$$

which, by putting $X = X + \frac{\pi}{2}$, is seen to be of the same form as (4), p. 195. It appears then that in this case no new types are obtained. As in all other cases, the transformation $X = iX'$, $Y = iY'$, $Z = iZ'$ will transform (5) into a form involving e^X instead of $\tan X$, viz.:

$$e^X = \frac{A + 2Z + B(Y^2 + Z^2)}{A - 2Z + B(Y^2 + Z^2)}.$$

THEOREM: *To a unicursal quartic with two cusps and one double point with a real or imaginary pair of tangents there correspond* ∞^1 *types of translation-surfaces that can be generated in four different ways. These surfaces may by proper transformations be brought into either of the two forms:*

$$\tan Z = \frac{A + B(X^2 + Y^2)}{2X},$$

$$e^Z = \frac{A' - 2X + B'(X^2 + Y^2)}{A' + 2X + B'(X^2 + Y^2)}.$$

We shall now collect the results obtained in the following table:

	THE PLANE (xy) AT INFINITY.	SPACE (X, Y, Z).
I.	a. Quartic curve with three double points having real tangents.	a. $A(1 - e^{X+Y+Z}) + B(e^{Y+Z} - e^X) + C(e^{X+Z} - e^Y) + D(e^{X+Y} - e^Z) = 0.$
	b. Quartics with three conjugate points.	b. $A \tan X \tan Y + B \tan X \tan Z + C \tan Y \tan Z + D = 0.$
	c. Quartics having three double points of which two are conjugate points.	c. $e^{2X} = \dfrac{L \tan X \tan Z + M \tan Y + N \tan Z + P}{L \tan X \tan Z - M \tan Y - N \tan Z + P}.$
	d. Quartics with three double points of which one is a conjugate point.	d. $\tan 2X = \dfrac{A(e^{Y+Z} - 1) + B(e^Y - e^Z)}{A'(e^{Y+Z} + 1) + B'(e^Y + e^Z)}.$
II.	a. Quartics having one cusp and two double points, both having distinct and real tangents.	a. $X = \dfrac{A(e^{Z+Y} - 1) + B(e^Z - e^Y)}{A'(e^{Z+Y} + 1) + B'(e^Z + e^Y)}.$
	b. Quartics having one cusp and two double points, one of which is a conjugate point.	b. $X = \dfrac{A \tan Z(e^Y + 1) + B(e^Y - 1)}{A' \tan Z(e^Y - 1) + B'(e^Y + 1)}.$
III.	a. Quartics having two cusps and one double point.	a. $\tan Z = \dfrac{A + B(X^2 + Y^2)}{2X}.$
	b. Quartics with two cusps and a conjugate point.	b. $e^Z = \dfrac{A' - 2X + B'(X^2 + Y^2)}{A' + 2X + B'(X^2 + Y^2)}.$

VIII.

Quartics with a Triple Point (Real Tangents).

A quartic with a triple point may be written $yu_3 = u_4$, where u_3 is homogeneous of the third degree in x and z and u_4 of the fourth degree in the same variables. If $y = 0$ be taken as a double tangent and $x = 0$, $z = 0$ be two of the tangents at the triple point, the curve will take the form

$$xyz(x - \alpha z) = (x^2 + k_1 xz + k_2 z^2)^2. \tag{1}$$

Putting now $\alpha z = z'$, $y = \alpha y'$ $x = x'$, this equation reduces to

$$xyz(x - z) = (x^2 + axz + bz^2)^2,$$

where $a = \frac{k_1}{\alpha}$, $b = \frac{k_2}{\alpha^2}$; or, in Cartesian coordinates,

$$xy(x - 1) = (x^2 + ax + b)^2. \tag{2}$$

The corresponding translation-surface may now be written:

$$\begin{aligned} X &= \int \frac{dx_1}{x_1 - 1} + \int \frac{dx_2}{x_2 - 1}, \\ Y &= \int \frac{(x_1^2 + ax_1 + b)^2}{x_1^2(x_1 - 1)^2} + \int \frac{(x_2^2 + ax_2 + b)^2}{x_2^2(x_2 - 1)^2}, \\ Z &= \int \frac{dx_1}{x_1(x_1 - 1)} + \int \frac{dx_2}{x_2(x_2 - 1)}, \end{aligned} \tag{3}$$

from which we derive the following equalities:

$$\left.\begin{aligned} &X = \log(x_1 - 1)(x_2 - 1), \quad Z = \log\frac{(x_1 - 1)(x_2 - 1)}{x_1 x_2}, \\ &x_1 + x_2 = e^{X+Z} - e^X + 1, \qquad x_1 x_2 = e^{X-Z}; \end{aligned}\right\} \tag{4}$$

$$\begin{aligned} Y = x_1 + x_2 + 2(a+1)X + 2bZ - (a+1)^2\left(\frac{1}{x_1 - 1} + \frac{1}{x_2 - 1}\right) - 2b(a+1)Z \\ -2b(a+1)\left(\frac{1}{x_1 - 1} + \frac{1}{x_2 - 1}\right) + b^2\left[-\frac{1}{x_1} - \frac{1}{x_2} + 2(X - Z) - \left(\frac{1}{x_1 - 1} + \frac{1}{x_2 - 1}\right) - 2X\right]; \end{aligned}$$

so that we have from (4)

$$\begin{aligned} Y - 2(a+1)X + 2b(a+b)Z = x_1 + x_2 - (a+b+1)^2\left[\frac{x_1 + x_2 - 2}{(x_1 - 1)(x_2 - 1)}\right] \\ - b^2\left(\frac{x_1 + x_2}{x_1 x_2}\right). \end{aligned} \tag{5}$$

Substituting from (4) and putting the left-hand side of (5) equal to a new Y, we have

$$Y = 1 + e^{X-Z} - e^X - (a+b+1)^2\left[\frac{e^{X-Z} - e^X - 1}{e^X}\right] - b^2\left[\frac{e^{X-Z} - e^X + 1}{e^{X-Z}}\right],$$

which may be reduced to the form

$$Y = e^{X-Z} - b^2 e^{Z-X} - \left[e^X - (a+b+1)^2 e^{-X}\right] + b^2\left[e^Z - \frac{(a+b+1)^2}{b^2} e^{-Z}\right].$$

Putting $X = X' + k_1$, $Y = Y'$, $Z = Z' + k_3$, we shall determine k_1 and k so as to make the surface symmetrical with respect to the origin; the center is easily found to be

$$k_1 = \log(a+b+1), \quad k_2 = 0, \quad k_3 = \frac{a+b+1}{b}, \quad (a+b+1 \neq 0),$$

so that we have, finally,

$$Y = b\left(e^{X-Z} - e^{Z-X}\right) - (a+b+1)\left(e^X - e^{-X}\right) + b(a+b+1)\left(e^Z - e^{-Z}\right), \tag{6}$$

which surface has two imaginary periods.

By means of the well-known transformation $X = iX'$, $Y = iY'$, $Z = iZ'$, it may be transformed into the form

$$Y = b \sin(X - Z) - (a+b+1)\sin X + b(a+b+1)\sin Z, \tag{6'}$$

which has two real periods.

1. If $a + b < -1$, the center of symmetry is imaginary.

2. If $a^2 - 4b = 0$, *the quartic has a point of undulation at* $x = -\frac{a}{2}$, $y = 0$; in this case the surface (6) becomes

$$Y = \frac{a^2}{4}\left(e^{X-Z} - e^{Z-X}\right) - \left(\frac{a}{2}+1\right)^2\left(e^X - e^{-X}\right) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2\left(e^Z - e^{-Z}\right), \tag{7}$$

or,

$$Y = \frac{a^2}{4}\sin(X - Z) - \left(\frac{a}{2}+1\right)^2 \sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2 \sin Z. \tag{7'}$$

THEOREM: I. *To a quartic having a triple point with real tangents correspond* ∞^2 *translation-surfaces of the form*

$$Y = b\left(e^{X-Z} - e^{Z-X}\right) - (a+b+1)\left(e^X - e^{-X}\right) + b(a+b+1)\left(e^Z - e^{-Z}\right), \tag{6}$$

or,

$$Y = b \sin(X - Z) - (a+b+1)\sin X + b(a+b+1)\sin Z. \tag{6'}$$

II. *To a quartic with a triple point and a point of undulation correspond* ∞^1 *translation-surfaces of the form*

$$Y=\frac{a^2}{4}(e^{X-Z}-e^{Z-X})-\left(\frac{a}{2}+1\right)^2(e^X-e^{-X})+\frac{a^2}{4}\left(\frac{a}{2}+1\right)^2(e^Z-e^{-Z}), \tag{7}$$

or,

$$Y=\frac{a^2}{4}\sin(X-Z)-\left(\frac{a}{2}+1\right)^2\sin X+\frac{a^2}{4}\left(\frac{a}{2}+1\right)^2\sin Z. \tag{7'}$$

IX.

Quartics Having a Triple Point with One Real and Two Imaginary Tangents.

We put the quartic in the form

$$yz(x^2+z^2)=(x^2+axz+bz^2)^2, \tag{1}$$

or, putting $z=1$,

$$y(x^2+1)=(x^2+ax+b)^2, \tag{2}$$

from which we derive the surface (x_1 and x_2 being the parameters):

$$\begin{aligned} X&=\log(x_1^2+1)(x_2^2+1), \qquad Z=\tan^{-1}x_1+\tan^{-1}x_2, \\ Y&=x_1+x_2+a\log(x_1^2+1)(x_2^2+1)+2(b-1)(\tan^{-1}x_1+\tan^{-1}x_2) \\ &\quad+a^2(\tan^{-1}x_1+\tan^{-1}x_2)-a(b-1)\left[\frac{1}{x_1^2+1}+\frac{1}{x_2^2+1}\right] \\ &\quad+\left[\frac{(b-1)^2-a^2}{2}\right]\left[\frac{x_1}{x_1^2+1}+\frac{x_2}{x_2^2+1}\right]+\left[\frac{(b-1)^2-a^2}{2}\right](\tan^{-1}x_1+\tan^{-1}x_2); \end{aligned} \tag{3}$$

so that we have

$$\begin{aligned} Y-2aX-[2(b-1)Z+a^2Z+\tfrac{1}{2}((b-1)^2-a^2)Z]&=\sin Z e^X \\ -a(b-1)[\sin^2 Z+2e^{-X}\cos Z]+\left[\frac{(b-1)^2-a^2}{2}\right]&\sin Z(2e^{-X}-\cos Z), \end{aligned}$$

or,

$$Y=e^Xe^Z+\frac{a(b-1)}{2}[\cos 2Z-4e^{-X}\cos Z]+\frac{(b-1)^2-a^2}{2}[\sin Z(2e^{-X}-\cos Z)].$$

Transforming to the center of symmetry, putting $X=X'+k_1$, $Y=Y'$, $Z=Z'+k_3$, we easily find

$$k_1=-\tfrac{1}{2}\log[a^2+(b-1)^2], \quad k_3=\tan^{-1}\frac{a}{b-1},$$

and the surface reduces to the form

$$Y=(b-1)\sin Z.(e^X+e^{-X})+a\cos Z.(e^X-e^{-X})-\tfrac{1}{4}[(b-1)^2+a^2]\sin 2Z.^* \tag{4}$$

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* Here, as elsewhere, we have omitted the somewhat long algebraic calculations. As a check we have used throughout the property of symmetry which characterises all these surfaces.

If the quartic in addition has a point of undulation, we have $a^2 = 4b$, so that (4) reduces to

$$Y = (b-1)\sin Z(e^X + e^{-X}) + 2\sqrt{b}\cos Z(e^X - e^{-X}) - \tfrac{1}{4}(b+1)^2 \sin 2Z.$$

X.

Quartics with a Triple Point, Two of the Tangents Being Coincident.

The quartic may be written

$$yxz^2 = (x^2 + azx + bz^2)^2,$$

in which a may be reduced to unity,

$$yxz^2 = (x^2 + zx + bz^2)^2,$$

or,

$$yx = (x^2 + x + b)^2.$$

The corresponding surface is

$$Y = \tfrac{1}{3}X^3 - bX(e^Z + e^{-Z}), \tag{1}$$

the center of symmetry being $k_1 = 0$, $k_2 = 0$, $k_3 = \log b$. If $1 - 4b = 0$, the quartic will have a point of undulation, in which case the surface (1) becomes

$$Y = \tfrac{1}{3}X^3 - \tfrac{1}{4}X(e^Z + e^{-Z}).$$

We may now express the results obtained in VIII, IX and X thus:

THEOREM: I. *To a quartic with a triple point, the tangents being all real, there correspond* ∞^2 *types of translation-surfaces of the form*

$$Y = b(e^{X-Z} - e^{Z-X}) - (a+b+1)(e^X - e^{-X}) + b(a+b+1)(e^Z - e^{-Z}), \tag{6}$$

or,

$$Y = b\sin(X-Z) - (a+b+1)\sin X + b(a+b+1)\sin Z.$$

II. *To a quartic with a triple point, having real tangents and also a point of undulation, correspond* ∞^1 *types of translation-surfaces of the form*

$$Y = \frac{a^2}{4}(e^{X-Z} - e^{Z-X}) - \left(\frac{a}{2}+1\right)^2(e^X - e^{-X}) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2(e^Z - e^{-Z}), \tag{7}$$

or,

$$Y = \frac{a^2}{4}\sin(X-Z) - \left(\frac{a}{2}+1\right)^2 \sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2 \sin Z.$$

III. *To a quartic with a triple point, one pair of whose tangents are imaginary, there correspond* ∞^2 *translation-surfaces of the form*

$$Y = (b-1)\sin Z\,.\,(e^X + e^{-X}) + a\cos Z\,.\,(e^X - e^{-X}) - \tfrac{1}{4}[(b-1)^2 + a^2]\sin 2Z.$$

If the quartic also has a point of undulation, the corresponding surface is (∞^1 *types*):

$$Y = (b-1)\sin Z\,(e^X + e^{-X}) + 2\sqrt{b}\cos Z\,(e^X - e^{-X}) - \tfrac{1}{4}(b+1)^2 \sin 2Z.$$

IV. *To a quartic with a triple point, two of whose tangents are coincident, there correspond* ∞^1 *types of surfaces*

$$Y = \tfrac{1}{3}X^3 - bX(e^Z + e^{-Z}).$$

V. *To a quartic which in addition to the triple point* (*two coincident tangents*) *also has a point of undulation there corresponds a single type of surfaces of the form*

$$Y = \tfrac{1}{3}X^3 - \tfrac{1}{4}X(e^Z + e^{-Z}).$$

The last two surfaces may also be put in the form

$$Y = -\tfrac{1}{3}X^3 - 2b\cos Z\,.\,X,$$
$$Y = -\tfrac{1}{3}X^3 - \tfrac{1}{2}X\cos Z.$$

VI. *If finally all the tangents at the triple point coincide, two types of algebraic surfaces are obtained which have been discussed in a former paper,* where the proof is given.*

We now give a résumé of the results obtained:

PLANE AT INFINITY.	SPACE (X, Y, Z).
I. Quartics with a triple point (real tangents).	a. $Y = b(e^{X-Z} - e^{Z-X}) - (a+b+1)(e^X - e^{-X}) + b(a+b+1)(e^Z - e^{-Z})$. b. $Y = b\sin(X-Z) - (a+b+1)\sin X + b(a+b+1)\sin Z$.
II. Quartics with a triple point and a point of undulation.	a. $Y = \frac{a^2}{4}(e^{X-Z} - e^{Z-X}) - \left(\frac{a}{2}+1\right)^2(e^X - e^{-X}) + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2(e^Z - e^{-Z})$. b. $Y = \frac{a^2}{4}\sin(X-Z) + \left(\frac{a}{2}+1\right)^2 \sin X + \frac{a^2}{4}\left(\frac{a}{2}+1\right)^2 \sin Z$.
III. Quartics having a triple point with one real and two imaginary tangents.	$Y = (b-1)\sin Z.(e^X + e^{-X}) + a\cos Z.(e^X - e^{-X}) - \frac{1}{4}[(b-1)^2 + a^2]\sin 2Z$.
IV. Quartics having a triple point with one real and two imaginary tangents and also a point of undulation.	$Y = (b-1)\sin Z(e^X + e^{-X}) + 2\sqrt{b}\cos Z(e^X - e^{-X}) - \frac{1}{4}(b+1)^2 \sin 2Z$.
V. Quartics with a triple point, two of the tangents being coincident.	$Y = \frac{1}{3}X^3 - bX(e^Z + e^{-Z})$.
VI. Quartics with a triple point, two of the tangents being coincident, and having also a point of undulation.	$Y = \frac{1}{3}X^3 - \frac{1}{4}X(e^Z + e^{-Z})$.
VII. Quartics with a triple point, all three tangents being coincident.	Algebraic surface.
VIII. Quartic with a triple point, coincident tangents, and a point of undulation.	Algebraic surface.

* *Am. Jour. of Math.*, vol. 29: On a Certain Class of Algebraic Translation-Surfaces, pp. 384–385.

Quartics with a Tac-Node and Double Point.

A quartic with a tac-node and a double point takes the following form after a proper projective transformation:

$$x^4 + cx^3y + dxy^2z + ay^2z^2 + bx^2yz = 0. \tag{1}$$

By means of an affinity transformation this curve may be thrown into the form $(z = 1)$:

$$x^4 + x^3y + x^2y + ay^2 + bxy^2 = 0,$$

which may be represented parametrically as follows:

$$x = -\frac{a + \rho + \rho^2}{\rho + b}, \quad y = \frac{(a + \rho + \rho^2)^2}{\rho(\rho + b)^2}.$$

We shall distinguish between 5 cases:

1. $a > \frac{1}{4}$, the tac-node is imaginary.
2. $a < \frac{1}{4}$, the tac-node is real.
3. $a = \frac{1}{4}$, ramphoid cusp and a node.
4. $b = 0$, tac-node and cusp (node either real or imaginary according as $a \lesseqgtr \frac{1}{4}$).
5. $a = \frac{1}{4}$, $b = 0$, ramphoid cusp and cusp.

The last case gives rise to an algebraic surface, as we have shown in a former paper, and will not be discussed here. We shall not go into the details of the calculations, only giving the chief results.

1. In the first case we obtain the surface

$$\begin{aligned} X &= \int \frac{d\rho_1}{a + \rho_1 + \rho_1^2} + \int \frac{d\rho_2}{a + \rho_2 + \rho_2^2}, \\ Y &= \log \frac{\rho_1 \rho_2}{(\rho_1 + b)(\rho_2 + b)}, \\ Z &= \int \frac{(\rho_1 + b)\, d\rho_1}{(a + \rho_1 + \rho_1^2)^2} + \int \frac{d\rho_2}{(a + \rho_2 + \rho_2^2)^2}, \end{aligned} \tag{2}$$

which, after integrating and transforming linearly, may be written $(a > \frac{1}{4})$:

$$\begin{aligned} X &= \tan^{-1} \frac{\rho_1 + \frac{1}{2}}{\sqrt{a - \frac{1}{4}}} + \tan^{-1} \frac{\rho_2 + \frac{1}{2}}{\sqrt{a - \frac{1}{4}}}, \\ Y &= \log \frac{\rho_1 \rho_2}{(\rho_1 + b)(\rho_2 + b)}, \\ Z &= \frac{\rho_1 + c}{a + \rho_1 + \rho_1^2} + \frac{\rho_2 + c}{a + \rho_2 + \rho_2^2}, \quad \left(c = \frac{b - 2a}{2b - 1}\right). \end{aligned} \tag{3}$$

Eliminating ρ_1 and ρ_2 from these equations we obtain a surface of the form

$$Z=\frac{A e^{2Y}\tan^2 X+B e^{2Y}+C\tan^2 X+D e^{Y}\tan^2 X+E e^{2Y}\tan X+F e^{Y}\tan X+G e^{Y}+H\tan X+I}{A' e^{2Y}\tan^2 X+B' e^{2Y}+C'\tan^2 X+D' e^{Y}\tan^2 X+E' e^{2Y}\tan X+F'' e^{Y}\tan X+G' e^{Y}+H'\tan X+I'}. \tag{4}$$

We shall not endeavor to find the center of symmetry, as it involves very long calculations.

2. In the second case we get the same form as (4), only, instead of $\tan X$, we must substitute e^X.

3. Putting $a=\frac{1}{4}$ in (3) we obtain the surface

$$X=\frac{1}{\rho_1+\frac{1}{2}}+\frac{1}{\rho_2+\frac{1}{2}},$$
$$Y=\log\frac{\rho_1\rho_2}{(\rho_1+b)(\rho_2+b)},$$
$$Z=\frac{4}{(\rho_1+\frac{1}{2})^2}+\frac{2b-1}{(\rho_1+\frac{1}{2})^3}+\frac{4}{(\rho_2+\frac{1}{2})^2}+\frac{2b-1}{(\rho_2+\frac{1}{2})^3},$$

from which, by elimination of ρ_1 and ρ_2 and tranforming to the center of symmetry, we obtain a surface of the form

$$e^Y=\frac{A+BX+CX^2+DX^3+EZ}{A-BX+CX^2-DX^3-EZ}. \tag{5}$$

4. When $b=0$, we have the curve

$$x^4+x^3y+x^2y+ay^2=0,$$

and the corresponding surface may be written:

$$\begin{aligned} X&=\tan^{-1}\frac{\rho_1+{}^{1}}{\sqrt{a-\frac{1}{4}}}+\tan^{-1}\frac{\rho_2+\frac{1}{2}}{\sqrt{a-\frac{1}{4}}},\\ Y&=\frac{1}{\rho_1}+\frac{1}{\rho_2},\\ Z&=\frac{\rho_1+2a}{a+\rho_1+\rho_1^2}+\frac{\rho_2+2a}{a+\rho_2+\rho_2^2}; \end{aligned} \tag{6}$$

and in case $a<\frac{1}{4}$:

$$\begin{aligned} X&=\log\frac{(\rho_1-\sqrt{\frac{1}{4}-a})(\rho_2-\sqrt{\frac{1}{4}-a})}{(\rho_1+\sqrt{\frac{1}{4}-a})(\rho_2+\sqrt{\frac{1}{4}-a})},\\ Y&=\frac{1}{\rho_1}+\frac{1}{\rho_2},\\ Z&=\frac{\rho_1+2a}{a+\rho_1+\rho_1^2}+\frac{\rho_2+2a}{a+\rho_2+\rho_2^2}. \end{aligned} \tag{7}$$

From (6) we obtain a surface of the form

$$Z=\frac{AY^2\tan^2X+BY^2+C\tan^2X+DY\tan^2X+EY^2\tan X+FY\tan X+GY+H\tan X+I}{A'Y^2\tan^2X+B'Y^2+C'\tan^2X+D'Y\tan^2X+E'Y^2\tan X+F'Y\tan X+G'Y+H'\tan X+I'},$$

and from (7) *a surface of the same form,* e^X *being substituted for tan* X.

Remark. If in the second case the node is imaginary, the curve may be put in a suitable form so that, in (5), tan Y will appear instead of e^Y.

Quartics with an Osc-Node.

This case has been discussed in my former paper on algebraic translation-surfaces* with a fourfold mode of representation. It was found that the surface may be represented parametrically as follows:

$$X=\int\frac{\left(\rho_1+\frac{c}{2}\right)d\rho_1}{\rho^4-\left(\frac{c^2}{4}-1\right)^2}+\int\frac{\left(\rho_2+\frac{c}{2}\right)d\rho_2}{\rho_2^4-\left(\frac{c^2}{4}-1\right)^2},$$

$$Y=\int\frac{d\rho_1}{\rho_1^2-\left(\frac{c^2}{4}-1\right)}+\int\frac{d\rho_2}{\rho_2^2-\left(\frac{c^2}{4}-1\right)},$$

$$Z=\int\frac{\left(c\rho_1+\frac{c^2}{2}-1\right)d\rho_1}{\left(\rho_1^2+\frac{c^2}{4}-1\right)^2\left[\rho_1^2-\left(\frac{c^2}{4}-1\right)\right]}+\int\frac{\left(c\rho_2+\frac{c^2}{2}-1\right)d\rho_2}{\left(\rho_2^2+\frac{c^2}{4}-1\right)\left[\rho_2^2-\left(\frac{c^2}{4}-1\right)\right]},$$

which we shall not discuss in detail. When $c=\pm 2$, we obtain an algebraic surface which has been treated in the previous paper. The classification of translation-surfaces connected with a unicursal quartic is thus completed. The cubic surface obtained on p. 178 deserves a closer study inasmuch as, from the standpoint of the theory of translation-surfaces, it holds a unique place in geometry. The existence and properties of such a surface fully realize the expectation of Georg Scheffers when he says (*Acta Math.*, Vol. XXVIII, 1904, p. 90): "Die grosse Zahl verschiedenartiger Typen von Translationsflächen, die sich aus dem Lie'schen Theorem ergeben, ist bisher, so viel ich weiss, noch nicht genauer untersucht worden, obgleich ihre Betrachtung wegen des innigen Zusammenhangs mit dem Abel'schen Theorem sowohl in geometrischer als auch in analytischer Hinsicht gewiss sehr lohnend sein würde."

* On a Certain Class, etc.: *Am. Jour. of Math.* vol. 29, p. 382.

The interpretation of this and the previous paper from the standpoint of the theory of functions is so evident that we have not thought it worth while to dwell on it. The functions obtained are all *degenerate Abelian Integrals of the second and third kind** (polar and logarithmic singularities).

UNIVERSITY OF WEST VIRGINIA, *July* 20, 1907.

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